UNIT-IV

SOME TWO-DIMENSIONAL FLOWS :-

Suppose a fluid moves in a way that at any given instant the flow pattern in a cartesian plane is the same as that in all other parellel planes with in the fluid.

$z=0$

Then the flows is said to be z-dimensional flow.

USE OF CYLINDRICAL POLAR CO-ORDINATES :-

If the flow is irrotational then the equation

$\nabla\phi$

satisfied by the velocity potential ϕ at any point having cylindrical polar co-ordinates (R, θ, z) is,

$$\frac{1}{R}\left[\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right) + \frac{\partial}{\partial\theta}\cdot\frac{1}{R}\left(\frac{\partial\phi}{\partial\theta}\right) + \right.$$

$$\frac{\partial}{\partial z}R\cdot\left(\frac{\partial\phi}{\partial z}\right) = 0.$$

$\nabla^2\phi = 0$

$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

$$\frac{1}{R}\left[R\frac{\partial^2\phi}{\partial R^2}+\frac{\partial\phi}{\partial R}\cdot 1+\frac{1}{R}\cdot\frac{\partial^2\phi}{\partial\theta^2}+R\frac{\partial^2\phi}{\partial z^2}\right]=0$$

$$\frac{\partial^2\phi}{\partial R^2}+\frac{1}{R}\cdot\frac{\partial\phi}{\partial R}+\frac{1}{R^2}\cdot\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0.$$

Multiply by R^2,

$$R^2\frac{\partial^2\phi}{\partial R^2}+R\frac{\partial\phi}{\partial R}+\frac{\partial^2\phi}{\partial\theta^2}+R^2\frac{\partial^2\phi}{\partial z^2}=0 \rightarrow (1)$$

Equation (1) represents laplace homogeneous equation.

In the two dimensional flow of cylindrical co-ordinates $\cancel{\phi(R,\theta)}$ $P(r,\theta)$,

(ie) $z=0$,

$$\frac{\partial^2\theta}{\partial z^2}=0.$$

Now consider,

$$\Rightarrow R^2\frac{\partial^2\phi}{\partial R^2}+R\frac{\partial\phi}{\partial R}+\frac{\partial^2\phi}{\partial\theta^2}=0 \rightarrow (2)$$

Now consider,

$$\phi(R,\theta) = f(R)\, g(\theta).$$

(ie) $\phi(R,\theta) = fg$

$$\left.\begin{aligned} \frac{\partial\phi}{\partial R} &= f'g \; ; \quad \frac{\partial\phi}{\partial\theta} = fg' \\ \frac{\partial^2\phi}{\partial R^2} &= f''g \; ; \quad \frac{\partial^2\phi}{\partial\theta^2} = fg'' \end{aligned}\right\} \rightarrow ③$$

sub ③ in ②,

$$R^2 \frac{f''(g)}{fg} + \frac{Rf'g}{fg} + \frac{fg''}{fg} = 0.$$

$\div fg$.

$$R^2 f''/f + \frac{Rf'}{f} + \frac{g''}{g} = 0.$$

$$\frac{R^2 f''}{f} + \frac{Rf'}{f} = -\frac{g''}{g} \rightarrow ④$$

The LHS of equation ④ is a function of R only.

The RHS of eqn ④ is a function of θ only.

Thus each is constant.

Let n^2 be the value of constant.

Thus,

$$R^2\frac{f''}{f} + \frac{Rf'}{f} = \frac{-g''}{g} = n^2 \text{ (say)}$$

$$\Rightarrow R^2\frac{f''}{f} + R\frac{f'}{f} = n^2$$

$$\Rightarrow R^2 f'' + Rf' = n^2 f.$$

$$\Rightarrow R^2 f'' + Rf' - n^2 f = 0 \rightarrow (5)$$

and

$$\frac{-g''}{g} = n^2$$

$$g'' = -n^2 g$$

$$g'' + n^2 g = 0 \rightarrow (6)$$

Let,

$$R = e^t$$

$$\frac{dR}{dt} = e^t.$$

$$\frac{df}{dR} = \frac{df}{dt}\cdot\frac{dt}{dR} \quad [\because dR/dt = e^t]$$

$$= \frac{df}{dt}\cdot e^{-t} \quad [dt/dR = e^{-t}]$$

$$\Rightarrow \frac{df}{dR} = \frac{df}{dt}\cdot\frac{1}{R} \quad [\because R = e^t,\ 1/R = e^{-t}]$$

$$\Rightarrow R\frac{df}{dR} = \frac{df}{dt}$$

$$\frac{d^2f}{dR^2} = d/dt\,(df/dR)\cdot dt/dR$$

$$= d/dt\,(df/dt\; e^{-t})\,e^{-t}$$

$$= e^{-2}\!\left(-df/dt\; e^{-t} + \frac{d^2f}{dt^2}\,e^{-t}\right)e^{-t}$$

$$= e^{-2t}\left(\frac{d^2f}{dt^2} - df/dt\right)$$

$$\frac{d^2f}{dR^2} = 1/R^2\left[\frac{d^2f}{dt^2} - \frac{df}{dt}\right].$$

$$R^2\frac{d^2f}{dR^2} = \frac{d^2f}{dt^2} - \frac{df}{dt}$$

$$R^2 f'' = \frac{d^2f}{dt^2} - \frac{df}{dt}.$$

(5) $\Rightarrow R^2f'' + Rf' - n^2f = 0$

$$\frac{d^2f}{dt^2} - \cancel{\frac{df}{dt}} + \cancel{\frac{df}{dt}} - n^2f = 0.$$

$$\frac{d^2f}{dt^2} - n^2f = 0 \rightarrow (7)$$

$$(D^2 - n^2)f = 0.$$

Auxillary equation,

$$m^2 - n^2 = 0$$

$$m = \pm n.$$

$$\therefore f = A_n e^{nt} + B_n e^{-nt} \Rightarrow A e^{\phi} + B e^{-\phi}.$$

$$f(R) = A_n e^{nt} + B_n e^{-nt}.$$

$$f(R) = A_n R^{n} + B_n R^{-n}.$$

$$(6) \Rightarrow g'' + g n^2 = 0$$

$$(D^2 + n^2) g = 0$$

Auxillary equation,

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm in.$$

$$g(\theta) = e^{\alpha x} [C_n \cos \beta x + D_n \sin \beta x].$$

$$g(\theta) = C_n \cos n\theta + D_n \sin n\theta.$$

$\therefore$ The soln of $\phi(R,\theta) = f(R) g(\theta)$.

$$\phi(R,\theta) = (A_n R^n + B_n R^{-n})(C_n \cos n\theta + D_n \sin n\theta)$$

$$\phi(R,\theta) = \sum_{n=1}^{\infty} (A_n R^n + B_n R^{-n}) \cdot (C_n \cos n\theta + D_n \sin n\theta)$$

This is general solution of $\phi(R,\theta)$.

case (i) :-

put $n = 1$

$$\phi(R,\theta) = (A_1 R^1 + B_1 R^{-1})(C_1 \cos\theta + D_1 \sin\theta)$$

$$(ie)\ \phi = (AR + B/R)(C\cos\theta + D\sin\theta)$$

$$\phi(R,\theta) = (R + 1/R)(\cos\theta + \sin\theta)$$

$$R\cos\theta + R\sin\theta + 1/R\cos\theta + 1/R\sin\theta$$

$\phi = R\cos\theta$, $\phi = R\sin\theta$, $\phi = 1/R\cos\theta$, $\phi = 1/R\sin\theta$ $\because$ These are harmonic.

case (ii) :-

put $n = 0$

$$(7) \Rightarrow \frac{d^2 f}{dt^2} - n^2 f = 0$$

$$\frac{d^2 f}{dt^2} = 0.$$

$$D^2 f = 0.$$

Auxillary equation :-

$$m^2 = 0$$

$$m = 0.$$

$$(A + Bx)e^{mx} = \frac{d^2 y}{dx^2}$$

$$f = (K_1 + K_2 t)e^{0t} = (A + Bx)e^{0x}$$

$$f = (K_1 + K_2 t)$$

$$\left[\because R = e^t,\ \log R = t\right]$$

$$f(R) = (K_1 + K_2 \log R).$$

put $n=0$

(6) $\Rightarrow g'' + n^2 g = 0$

$g'' = 0$

Auxillary equation,

$m = 0$.

$g = (K_3 + K_4\theta)e^{m\theta}$

$= (K_3 + K_4\theta)e^{0\theta}$

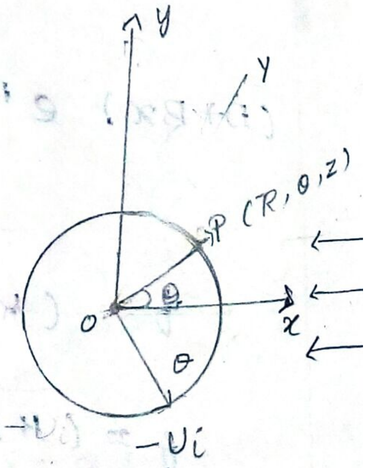
$g(\theta) = (K_3 + K_4\theta)$.

$\therefore$ The solution is,

$\phi(R,\theta) = f(R)g(\theta)$ for $m=0$

$\phi(R,\theta) = (K_1 + K_2 \log R)(K_3 + K_4\theta)$.

UNIFORM FLOW PAST A FIXED INFINITE CIRCULAR CYLINDER :-



The velocity of the uniform stream $= -U\hat{i}$.

Flowing past the fluid cylinder.

$R = a$ and $P(R,\theta,z)$ is a point.

in the fluid for cylindrical polar co-ordinates $P(x, y, z)$. The common co-ordinate x being Rebundant has the flow is two dimensional.

Velocity potential due to the uniform stream is,

$$Ux = UR\cos\theta.$$

$$\left\{\begin{array}{l} \because\ q = -\nabla\phi \Rightarrow -Ui = -\partial\phi/\partial x\ i \\ \because\ -U = -\partial\phi/\partial x \Rightarrow \phi = \phi(x, y) \\ \therefore\ u = \phi(x, y) \Rightarrow Ux = U\, R\cos\theta \end{array}\right\}$$

we consider the cylindrical flow than it will produce a perturbation of the flow.

condition

The perturbation must be such as to satisfy laplace eqn and to become vanishingly small for large R.

The simplest harmonic equation,

$$x = \frac{\cos\theta}{R} \Rightarrow 1/R \cos\theta$$

The velocity potential perturbation

$$\phi(R, \theta) = A\frac{\cos\theta}{R}$$

The total velocity potential.

$$\phi(R, \theta) = UR\cos\theta + \frac{A\cos\theta}{R} \rightarrow (1)$$

$$\frac{\partial \phi}{\partial R} = U\cos\theta - \frac{A\cos\theta}{R^2}$$

$$\frac{\partial \phi}{\partial R} = \left(U - A/R^2\right)\cos\theta :$$

$$\frac{\partial \phi}{\partial R} = 0 \qquad (\because R = a)$$

$$\left(U - A/a^2\right)\cos\theta = 0$$

$$U - A/a^2 = 0.$$

$$A = Ua^2.$$

eqn ① $\Rightarrow \phi(R,\theta) = UR\cos\theta + \frac{Ua^2\cos\theta}{R}$

$$= U\cos\theta\,(R + a^2R^{-1})$$

Hence the velocity components at P is

$$q_R = -\frac{\partial \phi}{\partial R}$$

$$= -U\cos\theta\,\{1 + a^2(-R^{-2})\}$$

$$= -U\cos\theta\,\{1 - (a/R)^2\} = -U\cos\theta\,\{1 - a^2/R^2\}$$

$$q_\theta = -1/R\ \partial\phi/\partial\theta.$$

$$= -1/R \quad [-U \sin\theta (R + a^2 \cdot R^{-1})]$$

$$= \frac{U \sin\theta}{R} \left[R + a^2/R \right].$$

$$q_\theta = U \sin\theta \left[1 + (a/R)^2 \right] . = U \sin\theta \, (1 + a^2/R^2).$$

$$q_z = -\frac{\partial \phi}{\partial z}$$

$$q_z = 0.$$

AS $R \to 0$, $q_R \to U \cos\theta$, $q_\theta \to U \sin\theta$.

(approxiamately).

Example :-

A cylinder of infinite length and nearly circular section moves through an infinite volume of liquid with a velocity U at right angles to its axis and in the direction of the x axis if its section is specified by the equation

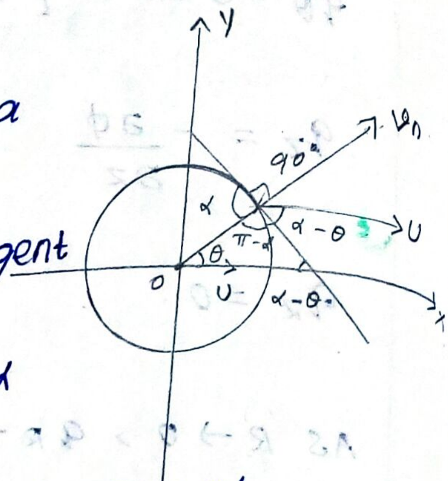
$$R = a (1 + \varepsilon \cos n\theta)$$

where, ε is small.

show that approxiamate value of the velocity potential is,

$$Ua\left\{ \frac{a}{R}\cos\theta + \varepsilon\left(\frac{a}{R}\right)^{n+1}\cos(n+1)\theta - \varepsilon\left(\frac{a}{R}\right)^{n-1}\cos(n-1)\theta \right\}$$

PROOF :-



we see from the figure a section of the cylinder, the tangent at the point P making angle α with the radius vector drawn from 'O'.

At large radial distance R from OZ, the fluid velocity becomes vanishingly small.

The suitable harmonic function R of the form,

$$R^{-K}\cos K\theta,\ R^{-K}\sin K\theta,\ K = 1, 2, \dots$$

Let us try the solution,

$$\phi(R,\theta) = \sum_{K=1}^{\infty} R^{-K}(A_K \cos K\theta + B_K \sin K\theta)$$

To find values of A_K and B_K,

At $\theta = 0$ on the boundary, $q_\theta = 0$.

which is satisby by taking $B_K = 0$, $K = 1, 2, \ldots$

Now, the function,

$$-\frac{1}{R}\sum_{K=1} R^{-K}(A_K - \sin K\theta \cdot K + B_K \cos K\theta \cdot K)$$

$$\phi(R, \theta) = \sum_{K=1}^{\infty} R^{-K} A_K \cos K\theta \rightarrow (1)$$

$$0 = -\frac{1}{R}(R^{-K} B_K \cdot K)$$

$$0 = -\frac{1}{R^{K+1}}(B_K \cdot K)$$

$$\Rightarrow B_K = 0$$

At any point (R, θ, z) in the fluid,

$$q_R = -\frac{\partial \phi}{\partial R}$$

$$= -\sum_{K=1}^{\infty} -K R^{-(K+1)} A_K \cos K\theta.$$

$$q_R = \sum_{K=1}^{\infty} K R^{-(K+1)} A_K \cos K\theta.$$

$$q_\theta = -\frac{1}{R}\, \partial\phi/\partial\theta$$

$$= -\frac{1}{R}\sum_{K=1}^{\infty} R^{-K} A_K(-\sin K\theta \cdot K)$$

$$= \sum_{K=1}^{\infty} \frac{R^{-K}}{R} A_K (\sin K\theta \cdot K).$$

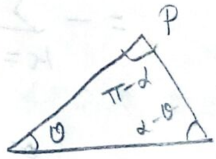
$$q_\theta = \sum_{K=1}^{\infty} R^{-(K+1)} \cdot K \cdot A_K \sin K\theta.$$

$$q_z = -\frac{\partial \phi}{\partial z} = 0.$$

At the point P on the surface, where $R = a(1+\varepsilon\cos n\theta)$.

$$\left.\begin{aligned} q_R &= \sum K a^{-(K+1)} (1+\varepsilon\cos n\theta)^{-(K+1)} A_K \cos K\theta \\ q_\theta &= \sum K a^{-(K+1)} (1+\varepsilon\cos n\theta)^{-(K+1)} A_K \sin K\theta \\ q_z &= 0. \end{aligned}\right\} \rightarrow (2)$$

From differential geometry 2-dimensional at P.



$$\cot(\pi-\alpha) = \frac{1}{R}\frac{dR}{d\theta}$$

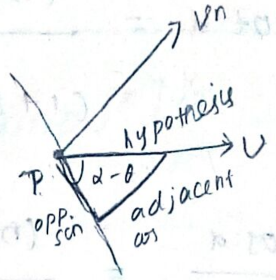
$$= d/d\theta\,(\log R)$$

$$= d/d\theta\,(\log[a(1+\varepsilon\cos n\theta)])$$

$$-\cot\alpha = \frac{1}{a(1+\varepsilon\cos n\theta)}\, a(-\varepsilon\sin n\theta)n$$

$$\cot\alpha = \frac{n\varepsilon\sin n\theta}{1+\varepsilon\cos n\theta}.$$

The normal component of velocity v_n of the cylinder at P is,



$$v_n = U \sin(\alpha - \theta)$$

$$v_n = U(\sin\alpha \cos\theta - \cos\alpha \sin\theta)$$

$$\because \cot^2\alpha = \frac{(n\varepsilon \sin n\theta)^2}{(1+\varepsilon \cos n\theta)^2}$$

$$1 + \cot^2\alpha = 1 + \frac{(n\varepsilon \sin n\theta)^2}{(1+\varepsilon \cos n\theta)^2}$$

$$\operatorname{cosec}^2\alpha = \frac{(1+\varepsilon \cos n\theta)^2 + (n\varepsilon \sin n\theta)^2}{(1+\varepsilon \cos n\theta)^2}$$

$$\frac{1}{\sin^2\alpha} = \frac{(1+\varepsilon \cos n\theta)^2 + (n\varepsilon \sin n\theta)^2}{(1+\varepsilon \cos n\theta)^2}$$

$$\sin^2\alpha = \frac{(1+\varepsilon \cos n\theta)^2}{(1+\varepsilon \cos n\theta)^2 + (n\varepsilon \sin n\theta)^2}$$

$$\sin\alpha = \frac{(1+\varepsilon \cos n\theta)}{\sqrt{(1+\varepsilon \cos n\theta)^2 + (n\varepsilon \sin n\theta)^2}}.$$

$$\because \cot\alpha = \frac{(n\varepsilon\sin n\theta)}{(1+\varepsilon\cos n\theta)}$$

$$\frac{\cos\alpha}{\sin\alpha} = \frac{(n\varepsilon\sin n\theta)}{(1+\varepsilon\cos n\theta)}$$

$$\cos\alpha = \frac{(n\varepsilon\sin n\theta)}{(1+\varepsilon\cos n\theta)}\sin\alpha$$

$$\cos\alpha = \frac{(n\varepsilon\sin n\theta)}{(1+\varepsilon\cos n\theta)} \cdot \frac{(1+\varepsilon\cos n\theta)}{\sqrt{(1+\varepsilon\cos n\theta)^2+(n\varepsilon\sin n\theta)^2}}$$

$$\therefore \cos\alpha = \frac{n\varepsilon\sin n\theta}{\sqrt{(1+\varepsilon\cos n\theta)^2+(n\varepsilon\sin n\theta)^2}}$$

$$v_n = v[\sin\alpha\cos\theta - \cos\alpha\sin\theta]$$

$$= v\left\{[\cos\theta]\left(\frac{(1+\varepsilon\cos n\theta)}{\sqrt{(1+\varepsilon\cos n\theta)^2+(n\varepsilon\sin n\theta)^2}}\right) - \sin\theta\left(\frac{n\varepsilon\sin n\theta}{\sqrt{(1+\varepsilon\cos n\theta)^2+(n\varepsilon\sin n\theta)^2}}\right)\right\}$$

$$v_n = \frac{U[\cos\theta(1+\varepsilon\cos n\theta) - \sin\theta(n\varepsilon\sin n\theta)]}{[(1+\varepsilon\cos n\theta)^2 + (n\varepsilon\sin n\theta)^2]^{1/2}} \rightarrow (3)$$

As there is no transport of fluid across the surface, the component of fluid velocity in the same direction at p must also be v_n.

(ie) $v_n = q_R \sin\alpha + q_\theta \cos\alpha \rightarrow (4)$

$$v_n = \frac{\sum_{K=1}^{\infty} KA_K a^{-(K+1)} \cos K\theta (1+\varepsilon\cos n\theta)^{-(K+1)} (1+\varepsilon\cos}{[(1+\varepsilon\cos n\theta)^2 + (n\varepsilon\sin n\theta)^2]^{1/2}}$$

$$+ \frac{\sum_{K+1}^{\infty} KA_K a^{-(K+1)} \sin K\theta (1+\varepsilon\cos n\theta)^{-(K+1)} n\varepsilon\sin n}{[(1+\varepsilon\cos n\theta)^2 + (n\varepsilon\sin n\theta)^2]^{1/2}}$$

$$v_n = \frac{\sum_{K=1}^{\infty} KA_K a^{-(K+1)} (1+\varepsilon\cos n\theta)^{-(K+1)}}{[(1+\varepsilon\cos n\theta)^2 + (n\varepsilon\sin n\theta)^2]^{1/2}}$$

$$\{\cos K\theta(1+\varepsilon\cos n\theta) + \sin K\theta \, n\varepsilon\sin n\theta\} \rightarrow (5)$$

Equating ③ and ⑤,

$$\Rightarrow \frac{U[\cos\theta\,(1+\varepsilon\cos n\theta) - \sin\theta\, n\varepsilon \sin n\theta]}{[(1+\varepsilon\cos n\theta)^2 + (n\varepsilon\sin n\theta)^2]^{1/2}}$$

$$= \frac{\sum_{K=1}^{\infty} K A_K\, a^{-(K+1)} (1+\varepsilon\cos n\theta)^{-(K+1)}}{[(1+\varepsilon\cos n\theta)^2 + (n\varepsilon\sin n\theta)^2]^{1/2}}$$

$$\{\cos K\theta\,(1+\varepsilon\cos n\theta) + \sin K\theta\, n\varepsilon\sin n\theta\}$$

$$\Rightarrow U[\cos\theta\,(1+\varepsilon\cos n\theta) - \sin\theta\, n\varepsilon\sin n\theta]$$

$$= \sum_{K=1}^{\infty} K A_K\, a^{-(K+1)} (1+\varepsilon\cos n\theta)^{-(K+1)} \{(\cos K\theta)$$

$$(1+\varepsilon\cos n\theta) + \sin K\theta\, n\varepsilon\sin n\theta\}$$

Approxiamating to the 1st order in ε,

$$\Rightarrow U\,[\cos\theta\,(1+\varepsilon\cos n\theta) - \sin\theta\, n\varepsilon\sin n\theta]$$

$$= \sum_{K=1}^{\infty} K A_K\, a^{-(K+1)} (1-(K+1)\varepsilon\cos n\theta)\{\cos K\theta$$

$$(1+\varepsilon\cos n\theta) + \sin K\theta\,(n\varepsilon\sin n\theta)\}$$

$\Rightarrow U[\cos\theta + \varepsilon\cos n\theta\cos\theta - n\varepsilon\sin n\theta\sin\theta]$

$$= \sum_{K=1}^{\infty} KA_K a^{-(K+1)}(1-\varepsilon(K+1)\cos n\theta)\{(\cos K\theta + \varepsilon\cos n\theta\cos K\theta) + \sin K\theta\, n\varepsilon\sin n\theta\}$$

$\Rightarrow U[\cos\theta + \varepsilon\cos n\theta\cos\theta - n\varepsilon\sin n\theta\sin\theta]$

$$= \sum_{K=1}^{\infty} KA_K a^{-(K+1)}\{\cos K\theta + \varepsilon\cos n\theta\cos K\theta + n\varepsilon\sin n\theta\sin K\theta - \varepsilon(K+1)\cos n\theta\cos K\theta - \varepsilon^2(K+1)\cos^2 n\theta\cos K\theta - n\varepsilon^2\sin n\theta\sin K\theta\,(K+1)\cos n\theta\}$$

$\Rightarrow U[\cos\theta + \varepsilon\cos n\theta\cos\theta - n\varepsilon\sin n\theta\sin\theta]$

$$= \sum_{K=1}^{\infty} KA_K a^{-(K+1)}\{\cos K\theta + \cancel{\varepsilon\cos n\theta\cos K\theta} + n\varepsilon\sin n\theta\sin K\theta - \varepsilon K\cos n\theta\cos K\theta - \cancel{\varepsilon\cos n\theta\cos K\theta}\}$$

(neglecting higher power of ε)

$\Rightarrow U[\underset{\cos a}{\cos\theta} + \varepsilon\underset{\cos b}{\cos n\theta}\cos\theta - n\varepsilon\underset{\sin a}{\sin n\theta}\underset{\sin b}{\sin\theta}]$

$$= \sum_{K=1}^{\infty} KA_K a^{-(K+1)}\{\cos K\theta + n\varepsilon\sin n\theta\sin K\theta - \varepsilon K\cos n\theta\cos K\theta\}$$

$\cos a \cos b = \dfrac{\cos(a-b) + \cos(a+b)}{2}$, $\sin a \sin b = \dfrac{\cos(a-b) - \cos(a+b)}{2}$

$\Rightarrow U\left[\cos\theta + \varepsilon/2 \; (\cos(n+1)\theta + \cos(n-1)\theta) - n\varepsilon/2\right.$

$a=n, b=1.$

$\left.[\cos(n-1)\theta - \cos(n+1)\theta)\right]$

$$= \sum_{k=1}^{\infty} kA_k \, a^{-(k+1)} \{\cos k\theta + n\varepsilon/2 \; (\cos(n-k)\theta$$

$$-\cos(n+k)\theta) - \varepsilon k/2 \; (\cos(n+k)\theta + \cos(n-k)\theta)\}$$

$\Rightarrow U[\cos\theta + \varepsilon/2 \; (\cos(n+1)\theta + \cos(n-1)\theta) - n\varepsilon/2$

$(\cos(n-1)\theta - \cos(n+1)\theta)$

$$\simeq \sum_{k=1}^{\infty} kA_k \, a^{-(k+1)} \{\cos k\theta + n\varepsilon/2 \cos(n-k)\theta$$

$$-n\varepsilon/2 \cos(n+k)\theta - k\varepsilon/2 \cos(n+k)\theta$$

$$-k\varepsilon/2 \cos(n-k)\theta\}$$

$\Rightarrow U\cos\theta + U\varepsilon/2 \; [\cos(n+1)\theta + \cos(n-1)\theta] - U\, n\varepsilon/2$

$[\cos(n-1)\theta - \cos(n+1)\theta]$

$$\simeq \sum_{k=1}^{\infty} kA_k \, a^{-(k+1)} \{\cos k\theta + \varepsilon/2 \, (n-k)\cos(n-k)\theta$$

$$-\varepsilon/2 \, (n+k)\cos(n+k)\theta\}$$